REAL ANALYSIS
TOPIC X - $[0,1]$ IS CONNECTED

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Definition 1. Let X be a topological space. We say that X is *connected* if there do not exist nonempty disjoint open sets whose union is X.

Let $I = [0,1]$ denote the closed unit interval in $\mathbb{R}$, endowed with the topology derived from the metric inherited as a subset of $\mathbb{R}$.

Proposition 1. *The topological space I is connected.*

Proof. Suppose, by way of contradiction, that there exist sets $U, V \subset I$ which are nonempty and open in I, such that $U \cup V = I$ and $U \cap V = \varnothing$. Without loss of generality, we may assume that $0 \in U$.

An open ball in I around 0 is of the form $[0, r)$ for some $r > 0$. Since U is open, there exists an open neighborhood of 0 of the form $[0, r)$ such that $[0, r) \subset U$.

Let $z = \sup\{x \in I \mid [0, x) \subset I\}$. Clearly, $z \geq r > 0$.

If $z = 1$, then $[0, 1) \subset U$, so either $V = \varnothing$ or $V = \{1\}$. But V is nonempty and open by assumption, and $\{1\}$ is not open, so this cannot be the case. Thus $z < 1$.

If $z \in U$, then z is an interior point of the open set U, so $(z - \delta, z + \delta) \subset U$ for some $\delta > 0$; but in that case, $[0, z + \delta) \subset U$, contradicting our definition of z as a supremum. Thus $z \notin U$. Hence $z \in V$.

Since V is open, z is an interior point of V, so $(z - \delta, z + \delta) \subset V$, for some $\delta > 0$. But then $(z - \delta, z) \subset U \cap V$, contradicting that U and V are disjoint. This contradiction completes the proof. □

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